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ORIGINAL ARTICLE

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Implementing process capability indices for a complete product

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Abstract This study develops a novel procedure for measuring the process capability indices of a complete product with several quality characteristics. The p-values of the estimators of C_{pl}, C_{pu}, and C_{pp}, so that the process works at least $100(1-\alpha)\%$ of the time, are provided. An evaluation checklist is also given to determine whether the process's potentiality and performance meet consumers' expectations.

Keywords Complete product · p-value · Process capability indices · Process yield · Unilateral and bilateral specifications

1 Introduction

Process capability indices have been widely applied in measuring product potential and performance. Many statisticians and quality engineers, such as Kane [1], Chan et al. [2], Choi and Owen [3], Boyles [4, 5], Pearn et al. [6], Kotz and Johnson [7], Spiring [8] and others have emphasized research into process capability indices to propose more effective methods of evaluating process potential and performance. A capability index is a dimensionless measure based on process parameters and specifications. It is generally in the form of (LSL, T, USL), where LSL is the lower specification limit, USL is the upper specification limit, and T is the target, and can be used to understand the effectiveness of the process simply and easily. The most commonly used process capability indices are,

$$C_p = \frac{USL-LSL}{6\sigma}\,, \tag{1}$$

$$C_{pl} = \frac{\mu-LSL}{3\sigma}\,, \tag{2}$$

$$C_{pu} = \frac{USL-\mu}{3\sigma}\,, \tag{3}$$

$$C_{pk} = \min\left\{\frac{USL-\mu}{3\sigma}, \frac{\mu-LSL}{3\sigma}\right\} = \min\{C_{pu}, C_{pl}\}\,, \tag{4}$$

where μ is process mean, and σ is process standard deviation (overall process variability). As noted by Boyles [4], C_p and C_{pk} are yield-based indices that are independent of the target T, and may thus fail to provide information on the variation about the target value.

Chan et al. [2] developed the index C_{pm}, based on process variation from the target under the loss function approach to quality improvement, where the denominator of C_{pm} refers to the expectation of Taguchi's loss function $E(X-T)^2 = \sigma^2 + (\mu-T)^2$. Thus, this index is defined as follows:

$$C_{pm} = \frac{USL-LSL}{6\sqrt{\sigma^2+(\mu-T)^2}}\,. \tag{5}$$

As stated by Chen [9, 10], the statistical properties of the natural estimator for index C_{pm} are analytically indeterminable. Accordingly, Greenwich and Jahr-Schaffrath [11] had introduced a new index C_{pp}, which is easier to use and analytically convenient. Let $D = d/3$; then the index C_{pp} is defined as

$$C_{pp} = \frac{(\mu-T)^2}{D^2} + \frac{\sigma^2}{D^2}\,, \tag{6}$$

where $D = \min\{USL-T, T-LSL\}$. Let $(\mu-T)^2/D^2$ be represented as C_{ia} (inaccuracy index) and σ^2/D^2 be represented as C_{ip} (imprecision index). Thus, $C_{pp} = C_{ia} + C_{ip}$ and is a simple transformation of the index C_{pm}.

C_{pp} is lower for a process that is more able to meet its specifications. Any non-zero value of C_{pp} indicates some degree of

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incapability in the process. These sub-indices also provide the fractions of the process incapability that are associated with the departure of the process mean from the target as well as to the process variation. This index is used to evaluate the process capability with bilateral specifications and suitable for processes of the "nominal-the-best" type.

For example, consider the following three processes A, B, and C with $\mu_A = m$, $\mu_B = m + d/6$, $\mu_C = m + \sqrt{3}d/6$ and $\sigma_A = d/3$, $\sigma_B = d/6$, $\sigma_C = d/6$ for $(LSL, T, USL) = (10, 13, 16)$; the value of C_{pp} is unity for each of A, B and C, and hence C_{pp} fails to distinguish between on-target and off-target processes. However, the values of (C_{ia}, C_{ip}) are (0, 1) for A, (1/4, 3/4) for B and (3/4, 1/4) for C. These processes are clearly distinguished by the separated information. The separated information can be used to distinguish processes with a highly conforming output (*COP*) from those with a less conforming output.

Some statisticians have explored several unilateral and bilateral specifications of process quality characteristics of multivariate process including, for example, Taam et al. [12], Chen [13], Boyles [14] and others. All of these indices have been designed to measure the process capability of a particular single quality characteristic of a complete product. Practically, most products have several important quality characteristics, each of which must fall within specifications to satisfy the customer (see Bothe [15]). Unfortunately, both univariate and multivariate process capability indices fail to achieve this aim. Bothe [15] points out that in a product capability study, the probability of each characteristic for a given product must first be determined to be within specifications, and then the combined probability that all characteristics are within specifications is calculated. This probability can be converted into the value of a capability index. Thus, various products can be evaluated along with the quality of the process in terms of several characteristics of a particular product.

Other indices such as C_{pl} and C_{pu} have been designed especially for processes with unilateral specifications (which require only an upper or lower specification limit). The index C_{pu} is suited to processes of the "smaller-the-better" type, whereas C_{pl} is suited to processes of the "larger-the-better" type.

This paper develops a novel procedure for measuring the process capability indices of a complete product with several characteristics. Also, the relationship between process yield and process capability index is explored, and the p-value of the estimators of some capability indices is presented. A checklist is constructed for determining whether the process's potentiality and performance meet consumers' expectations. The statistical properties of the proposed estimator associated with those indices are investigated under the process is normally distributed and statistically controlled.

2 Process yield

Suppose that the process distribution X is normally distributed as $N(\mu, \sigma^2)$. Let p_L and p_U denote the process yield (%Yield) of the lower and upper proportions, respectively, and be defined by,

$$\begin{aligned} p_L = \%\text{Yield} &= P(X > LSL) = 1 - F_X(LSL) \\ &= 1 - \Phi(-3C_{pl}) = \Phi(3C_{pl}) \,, \end{aligned} \tag{7}$$

$$p_U = \%\text{Yield} = P(X < USL) = F_X(USL) = \Phi(3C_{pu}) \,, \tag{8}$$

where $\Phi(x)$ is the cumulative distribution function of the standard normal distribution $N(0, 1)$.

Inverting the cumulative distribution function Eq. 7 and 8 yields,

$$C_{pl} = (1/3)\Phi^{-1}(p_L) \,, \tag{9}$$

$$C_{pu} = (1/3)\Phi^{-1}(p_U) \,. \tag{10}$$

Thus, C_{pl} and C_{pu} exactly measure the potential process yield.

Subtracting Eq. 7 from Eq. 8 gives the process yield (%Yield) of the bilateral specifications, under the assumption of normality:

$$\%\text{Yield} = F_X(USL) - F_X(LSL) = \Phi(3C_{pu}) - \Phi(3C_{pl}) \,. \tag{11}$$

When $\mu = m = (USL + LSL)/2$, the right hand side of Eq. 11, as a function of C_p, becomes,

$$\%\text{Yield} = 2\Phi(3C_p) - 1 \,, \tag{12}$$

which exactly measures (has a one-to-one correspondence with) the actual %Yield. Otherwise, the index C_p and %Yield do not have a one to one relationship.

Similarly, Boyles [4] showed that C_{pk} approximately measures actual process yield in form of the bilateral specifications, and is given by,

$$2\Phi(3C_{pk}) - 1 \le \%\text{Yield} < \Phi(3C_{pk}) \,. \tag{13}$$

Greenwich and Jahr-Schaffrath [11] introduced the incapability index C_{pp}, which purely separates information concerning the accuracy of a process from information concerning its precision. For $C_{pp} = c$ and $T = m$, the process yield is given by,

$$\begin{aligned} \%\text{Yield} = \Phi&\left(\frac{1 + \sqrt{c/9 - (\sigma/d)^2}}{\sigma/d}\right) \\ &+ \Phi\left(\frac{1 - \sqrt{c/9 - (\sigma/d)^2}}{\sigma/d}\right) - 1 \,, \end{aligned} \tag{14}$$

where $\sigma/d \le \sqrt{c}/3$. When $\sigma/d = \sqrt{c}/3$, such that $\mu = T$, $\%\text{Yield} = 2\Phi(3/\sqrt{c}) - 1$. For $c \le 1$, $\%\text{Yield} \ge 2\Phi(3/\sqrt{c}) - 1$ as shown in Table 1. A process is generally considered inadequate if $C_{pp} > 1$ and capable if $C_{pp} \le 1$. That is, if $C_{pp} \le 1$, and the process is perfectly centered, then %Yield can be expressed alternatively as,

$$\%\text{Yield} \ge 2\Phi(3) - 1 = 99.73\% \,, \tag{15}$$

and the deviation of the process from the target is $|\mu - T| < (d/3)$, for $c = 1$.

Table 1. %yield value corresponding to C_{pp}, for $\sigma/d = h\sqrt{c}/30$, $h = 1(1)\ 10$

h	$C_{pp}=1.0$	$C_{pp}=0.8$	$C_{pp}=0.6$	$C_{pp}=0.5$	$C_{pp}=0.4$	$C_{pp}=0.2$	$C_{pp}=0.1$
1	1.0000000000	1.0000000000	1.0000000000	1.0000000000	1.0000000000	1.000000000000	1.000000000000
2	1.0000000000	1.0000000000	1.0000000000	1.0000000000	1.0000000000	1.000000000000	1.000000000000
3	1.0000000000	1.0000000000	1.0000000000	1.0000000000	1.0000000000	1.000000000000	1.000000000000
4	0.9999999047	0.9999999994	1.0000000000	1.0000000000	1.0000000000	1.000000000000	1.000000000000
5	0.9999901293	0.9999996752	0.9999999991	1.0000000000	1.0000000000	1.000000000000	1.000000000000
6	0.9998771009	0.9999896260	0.9999998483	0.9999999952	1.0000000000	1.000000000000	1.000000000000
7	0.9994535480	0.9999187941	0.9999967955	0.9999997677	0.9999999957	1.000000000000	1.000000000000
8	0.9986466320	0.9997115194	0.9999785226	0.9999973562	0.9999998884	1.000000000000	1.000000000000
9	0.9977397829	0.9993947063	0.9999321329	0.9999881927	0.9999991445	0.999999999998	1.000000000000
10	0.9973000656	0.9992036603	0.9998924478	0.9999778949	0.9999978960	0.999999999980	1.000000000000

Table 2. k quality characteristics, random sample and estimators of *PCI*s

PCI	Specification *LSL*	*USL*	Sample data $1\ldots j\ldots n$	Sample mean	Sample *STD*	Estimator
C_{pl1}	L_{X1}		$X_{11}\ldots X_{1j}\ldots X_{1n}$	$\bar{X}_1$	S_{X1}	$\hat{C}_{pl1}$
$\vdots$	$\vdots$		$\vdots\quad\vdots\quad\vdots$	$\vdots$	$\vdots$	$\vdots$
C_{pli}	L_{Xi}		$X_{i1}\ldots X_{ij}\ldots X_{in}$	$\bar{X}_i$	S_{Xi}	$\hat{C}_{pli}$
$\vdots$	$\vdots$		$\vdots\quad\vdots\quad\vdots$	$\vdots$	$\vdots$	$\vdots$
C_{pla}	L_{Xa}		$X_{a1}\ldots X_{aj}\ldots X_{an}$	$\bar{X}_a$	S_{Xa}	$\hat{C}_{pla}$
C_{pu1}		U_{Y1}	$Y_{11}\ldots Y_{1j}\ldots Y_{1n}$	$\bar{Y}_1$	S_{Y1}	$\hat{C}_{pu1}$
$\vdots$		$\vdots$	$\vdots\quad\vdots\quad\vdots$	$\vdots$	$\vdots$	$\vdots$
C_{pui}		U_{Yi}	$Y_{i1}\ldots Y_{ij}\ldots Y_{in}$	$\bar{Y}_i$	S_{Yi}	$\hat{C}_{pui}$
$\vdots$		$\vdots$	$\vdots\quad\vdots\quad\vdots$	$\vdots$	$\vdots$	$\vdots$
C_{pub}		U_{Yb}	$Y_{b1}\ldots Y_{bj}\ldots Y_{bn}$	$\bar{Y}_b$	S_{Yb}	$\hat{C}_{pub}$
C_{pp1}	L_{Z1}	U_{Z1}	$Z_{11}\ldots Z_{1j}\ldots Z_{1n}$	$\bar{Z}_1$	S_{Z1}	$\hat{C}_{pp1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots\quad\vdots\quad\vdots$	$\vdots$	$\vdots$	$\vdots$
C_{ppi}	L_{Zi}	U_{Zi}	$Z_{i1}\ldots Z_{ij}\ldots Z_{in}$	$\bar{Z}_i$	S_{Zi}	$\hat{C}_{ppi}$
$\vdots$	$\vdots$	$\vdots$	$\vdots\quad\vdots\quad\vdots$	$\vdots$	$\vdots$	$\vdots$
C_{ppc}	L_{Zc}	U_{Zc}	$Z_{c1}\ldots Z_{cj}\ldots Z_{cn}$	$\bar{Z}_c$	S_{Zc}	$\hat{C}_{ppc}$

Clearly, when C_{pp} reaches the process-capable requirement, it not only reduces the expected process loss and clearly distinguishes the separated information, but also gives a higher %Yield. Hence, this paper uses C_{pp} to evaluate the quality characteristics with the bilateral specifications. Indices C_{pl}, and C_{pu} are also considered to evaluate quality characteristics with unilateral specification.

3 Capability index for a complete product

Assume that $k\ (= a+b+c)$ quality characteristics of a complete product are specified in three ways. C_{pl}, C_{pu} and C_{pp} are three indices to evaluate the process capabilities. There are a larger-the-better processes evaluated by C_{pli} $(i = 1, 2, \ldots, a)$, b smaller-the-better processes evaluated by C_{pui} $(i = 1, 2, \ldots, b)$, and c nominal-the-best processes evaluated by C_{ppi} $(i = 1, 2, \ldots, c)$. Let X_{ij}, $i = 1, 2, \ldots, a$, $j = 1, 2, \ldots, n$, denote the random samples of size n taken from larger-the-better processes and their mean and standard deviation are denoted by μ_{Xi} and σ_{Xi}, respectively; Y_{ij}, $i = 1, 2, \ldots, b$, $j = 1, 2, \ldots, n$, denote the random samples of size n taken from smaller-the-better processes and their mean and standard deviation are denoted by μ_{Yi} and σ_{Yi}, respectively; Z_{ij}, $i = 1, 2, \ldots, c$, $j = 1, 2, \ldots, n$, denote the random samples of size n taken from nominal-the-best processes with mean μ_{Zi} and standard deviation σ_{Zi}. The related notations are summarized in Table 2.

In Table 2 $\bar{X}_i$ and $S_{Xi} = (\sum_{j=1}^{n}(X_{ij} - \bar{X}_i)^2/(n-1))^{1/2}$, $i = 1, 2, \ldots, a$, are reasonable estimators of μ_{Xi} and σ_{Xi} for lower specifications; $\bar{Y}_i$ and $S_{Yi} = (\sum_{j=1}^{n}(Y_{ij} - \bar{Y}_i)^2/(n-1))^{1/2}$, $i = 1, 2, \ldots, b$, denote the natural estimators of μ_{Yi} and σ_{Yi} for the upper specifications; $\bar{Z}_i$ and $S_{Zi} = (\sum_{j=1}^{n}(Z_{ij} - \bar{Z}_i)^2/$

$(n-1))^{1/2}$, $i = 1, 2, \dots, c$, denote the natural estimators of μ_{Zi} and σ_{Zi} for the two-sided specification. Estimators of the three indices C_{pli}, C_{pui} and C_{ppi} are proposed as follows.

$$\hat{C}_{pli} = (\hat{X}_i - L_{Xi})/(3S_{Xi}), \quad i = 1, 2, \dots, a, \tag{16}$$

$$\hat{C}_{pui} = (U_{Yi} - \bar{Y}_i)/(3SYi), \quad i = 1, 2, \dots, b, \tag{17}$$

$$\hat{C}_{ppi} = \frac{(\hat{Z}_i - T_i)^2}{D_i^2} + \frac{S_{Zi}^2}{D_i^2}, \quad i = 1, 2, \dots, c, \tag{18}$$

where $D_i = \min\{U_{Zi} - T_i, T_i - L_{Zi}\}/3$.

When a complete product has three independent quality characteristics, the capability value of each of which must equal one, then given process yield %Yield $\geq \Phi(3PCI)$, %Yield of the three quality characteristics is at least 99.73%. Moreover, assuming independence, the process yields is 99.19% for a complete product. Therefore, when PCI ($= C_{pli}$ or C_{pui}) for the complete product is required to be one, the capability value of each characteristic must exceed one.

In fact, when the process yield of a complete product is required to be at least p, the independent quality characteristic process yields must be at least $1-(1-p)/k$ to satisfy consumer expectations. If this quality is not acceptable, then the process must be improved. The relationship between the process yield (p_i) for each quality characteristic and an acceptable capability index (PCI) is given by,

$$p_i = \Phi(3PCI). \tag{19}$$

When $PCI = c_{01}$, then

$$c_{01} = (1/3)\Phi^{-1}(p_i). \tag{20}$$

Thus, the corresponding process yield (p_i) and the preset value c_{01} are obtained for any quality characteristic of PCI, as presented in Table 3.

Assume that p_i is the ith quality characteristic of the process yield, $i = 1, 2, \dots, k$, and the quality of a complete product must be at least p; then, a complete product process yield satisfies,

$$\prod_{i=1}^{k} = p_i \geq p. \tag{21}$$

If $p_1 = p_2 = \dots = p_k$, then the process yield (p_i) of each characteristic is,

$$p_i = p^{1/k}, \quad i = 1, 2, \dots, k. \tag{22}$$

According to Eq. 20, the preset value c_{01} is obtained as follows:

$$c_{01} = (1/3)\Phi^{-1}\left(\sqrt[k]{p}\right). \tag{23}$$

Equation 23 yields the preset value, c_{01}, of the capability index of each independent characteristic presented in Table 5.

Similarly, C_{ppi} of each characteristic must be below 1.0. Therefore, the relationships among the process yield (p_i) of a complete product and an acceptable capability index C_{pp} is

Table 3. Value of p_i for various values of c_{01} for C_{pl} and C_{pu}

c_{01}	p_i	c_{01}	p_i	c_{01}	p_i
1.00	0.9986500328	1.34	0.9999708869	1.68	0.9999997669
1.01	0.9987771621	1.35	0.9999743783	1.69	0.9999998008
1.02	0.9988932462	1.36	0.9999774703	1.70	0.9999998299
1.03	0.9989991492	1.37	0.9999802062	1.71	0.9999998549
1.04	0.9990956774	1.38	0.9999826247	1.72	0.9999998763
1.05	0.9991835813	1.39	0.9999847609	1.73	0.9999998947
1.06	0.9992635596	1.40	0.9999866459	1.74	0.9999999104
1.07	0.9993362615	1.41	0.9999883078	1.75	0.9999999238
1.08	0.9994022895	1.42	0.9999897717	1.76	0.9999999353
1.09	0.9994622023	1.43	0.9999910600	1.77	0.9999999451
1.10	0.9995165175	1.44	0.9999921928	1.78	0.9999999534
1.11	0.9995657137	1.45	0.9999931879	1.79	0.9999999605
1.12	0.9996102333	1.46	0.9999940613	1.80	0.9999999666
1.13	0.9996504846	1.47	0.9999948272	1.81	0.9999999718
1.14	0.9996868442	1.48	0.9999954982	1.82	0.9999999761
1.15	0.9997196588	1.49	0.9999960855	1.83	0.9999999799
1.16	0.9997492474	1.50	0.9999965992	1.84	0.9999999830
1.17	0.9997759031	1.51	0.9999970480	1.85	0.9999999857
1.18	0.9997998951	1.52	0.9999974398	1.86	0.9999999879
1.19	0.9998214701	1.53	0.9999977815	1.87	0.9999999899
1.20	0.9998408543	1.54	0.9999980793	1.88	0.9999999915
1.21	0.9998582543	1.55	0.9999983385	1.89	0.9999999928
1.22	0.9998738594	1.56	0.9999985640	1.90	0.9999999940
1.23	0.9998878419	1.57	0.9999987600	1.91	0.9999999950
1.24	0.9999003595	1.58	0.9999989301	1.92	0.9999999958
1.25	0.9999115554	1.59	0.9999990777	1.93	0.9999999965
1.26	0.9999215603	1.60	0.9999992056	1.94	0.9999999970
1.27	0.9999304928	1.61	0.9999993164	1.95	0.9999999975
1.28	0.9999384606	1.62	0.9999994123	1.96	0.9999999979
1.29	0.9999455617	1.63	0.9999994951	1.97	0.9999999983
1.30	0.9999518845	1.64	0.9999995666	1.98	0.9999999986
1.31	0.9999575093	1.65	0.9999996284	1.99	0.9999999988
1.32	0.9999625087	1.66	0.9999996816	2.00	0.9999999990
1.33	0.9999669482	1.67	0.9999997274	2.01	0.9999999992

$$p_i = 2\Phi\left(3/\sqrt{C_{pp}}\right) - 1. \tag{24}$$

When $C_{pp} = c_{02} \leq 1$

$$c_{02} = \left(3/\Phi^{-1}\left(\left(\sqrt[k]{p}+1\right)/2\right)\right)^2 = \left(3/\Phi^{-1}\left((p_i+1)/2\right)\right)^2. \tag{25}$$

Tables 4 an 5 show the preset value c_{02} of the capability index for each independent characteristic of C_{pp}, obtained from Eq. 25.

4 Evaluating process performance

Assume that most quality characteristics of a product meet two-sided, lower, and upper specifications, each with a, b and c characteristics, respectively. The indices C_{pl} and C_{pu} are suited to evaluating quality characteristics with reference to the lower and upper specifications, respectively, and the index C_{pp} is suited to evaluating quality characteristics of the bilateral specifications.

Table 4. Value of p_i for various values of c_{02} for C_{pp}

c_{01}	p_i	c_{02}	p_i	c_{02}	p_i	c_{02}	p_i	c_{02}	p_i
0.01	1.000000000000000	0.21	0.9999999999	0.41	0.9999972001	0.61	0.9998774747	0.81	0.9991417669
0.02	1.000000000000000	0.22	0.9999999998	0.42	0.9999963235	0.62	0.9998609877	0.82	0.9990766961
0.03	1.000000000000000	0.23	0.9999999996	0.43	0.9999952320	0.63	0.9998428966	0.83	0.9990083830
0.04	1.000000000000000	0.24	0.9999999991	0.44	0.9999938878	0.64	0.9998231108	0.84	0.9989367649
0.05	1.000000000000000	0.25	0.9999999980	0.45	0.9999922489	0.65	0.9998015394	0.85	0.9988617820
0.06	1.000000000000000	0.26	0.9999999960	0.46	0.9999902695	0.66	0.9997780916	0.86	0.9987833765
0.07	1.000000000000000	0.27	0.9999999922	0.47	0.9999879000	0.67	0.9997526766	0.87	0.9987014934
0.08	1.000000000000000	0.28	0.9999999856	0.48	0.9999850866	0.68	0.9997252042	0.88	0.9986160803
0.09	1.000000000000000	0.29	0.9999999746	0.49	0.9999817719	0.69	0.9996955850	0.89	0.9985270870
0.10	1.000000000000000	0.30	0.9999999567	0.50	0.9999778949	0.70	0.9996637302	0.90	0.9984344660
0.11	1.000000000000000	0.31	0.9999999287	0.51	0.9999733907	0.71	0.9996295522	0.91	0.9983381724
0.12	1.000000000000000	0.32	0.9999998860	0.52	0.9999681915	0.72	0.9995929646	0.92	0.9982381637
0.13	1.000000000000000	0.33	0.9999998230	0.53	0.9999622258	0.73	0.9995538822	0.93	0.9981343999
0.14	0.999999999999999	0.34	0.9999997319	0.54	0.9999554193	0.74	0.9995122215	0.94	0.9980268436
0.15	0.999999999999990	0.35	0.9999996034	0.55	0.9999476952	0.75	0.9994679007	0.95	0.9979154596
0.16	0.999999999999936	0.36	0.9999994258	0.56	0.9999389737	0.76	0.9994208396	0.96	0.9978002153
0.17	0.999999999999654	0.37	0.9999991848	0.57	0.9999291730	0.77	0.9993709602	0.97	0.9976810805
0.18	0.999999999998452	0.38	0.9999988633	0.58	0.9999182091	0.78	0.9993181861	0.98	0.9975580272
0.19	0.999999999994082	0.39	0.9999984415	0.59	0.9999059965	0.79	0.9992624435	0.99	0.9974310300
0.20	0.999999999980187	0.40	0.9999978960	0.60	0.9998924478	0.80	0.9992036603	1.00	0.9973000656

Table 5. The capability index, c_{01} for C_{pl} or C_{pu} and c_{02} for C_{pp}, of each independent characteristic for given p and k

k	$p = 0.997300$		$p = 0.999993$		$p = 0.999999$	
	c_{01}	c_{02}	c_{01}	c_{02}	c_{01}	c_{02}
1	0.9274	1.0000	1.4467	0.4454	1.5847	0.3757
2	0.9999	0.8761	1.5025	0.4180	1.6307	0.3558
3	1.0404	0.8165	1.5274	0.4033	1.6572	0.3451
4	1.0683	0.7788	1.5398	0.3935	1.6757	0.3380
5	1.0896	0.7518	1.5646	0.3862	1.6898	0.3336
6	1.1066	0.7311	1.5895	0.3807	1.7014	0.3284
7	1.1209	0.7143	1.5895	0.3757	1.7111	0.3246
8	1.1331	0.7005	1.5895	0.3719	1.7194	0.3218
9	1.1438	0.6886	1.5895	0.3684	1.7269	0.3193
10	1.1533	0.6784	1.5895	0.3652	1.7334	0.3167
11	1.1618	0.6694	1.6094	0.3626	1.7392	0.3146
12	1.1696	0.6613	1.6190	0.3604	1.7447	0.3128
13	1.1766	0.6540	1.6253	0.3580	1.7495	0.3113
14	1.1832	0.6475	1.6013	0.3558	1.7542	0.3100
15	1.1892	0.6415	1.6111	0.3540	1.7583	0.3083

However, these indices include unknown parameters μ and σ, and can be estimated as described in Sect. 3.

Only $P\hat{C}I(\hat{C}_{pli}, \hat{C}_{pui})$ and $\hat{C}_{ppi}$ need be calculated and compared with the critical value required to assess the capability of a process. As stated above, simply considering the values of *PCI*(C_{pu}, C_{pl}) and C_{pp} determined from the sample data and then deciding upon whether the process is highly unreliable requires a simple procedure by which practitioners can determine correctly whether the process meets the required capability.

The following statistical hypotheses for *PCI* are considered to determine whether a given process achieves the required quality under the quality conditions. The process is of the required quality if $PCI > c_0$, and is not if $PCI < c_0$. The typically used benchmark values, c_0, including for example the preset value of *PCI* in Tables 3 or 5, as stated above, are selected to test the following hypotheses.

$$\begin{aligned} &H_0 : \text{process is not capable,} \\ &H_1 : \text{process is capable,} \end{aligned} \tag{26}$$

or equivalent to testing

$$\begin{aligned} &H_0 : PCI_i \le c_0 \,, \\ &H_1 : PCI_i > c_0 \,. \end{aligned} \tag{27}$$

The sampling distribution of $\hat{C}_{pli}$ or $\hat{C}_{pui}$ is part of the non-central t distribution with non-centrality parameter $\lambda_{1i} = 3\sqrt{n}C_{pli}$ and $\lambda_{2i} = 3\sqrt{n}C_{pui}$ and with $n-1$ degrees of freedom. The rejection probability, called the p-value, can be determined to make a decision.

C_{pli} and C_{pui} are estimated from the sample, such that $w_{pli} = \hat{C}_{pli}$ and $w_{pui} = \hat{C}_{pui}$. Then,

$$\begin{aligned} p\text{-value} &= P\left(\hat{C}_{pli} > w_{pli} | C_{pli} = c_{01}\right) \\ &= P\left(t'_{n-1}(\delta_{1i}) > 3\sqrt{n} w_{pli}\right) , \quad \delta_{1i} = 3\sqrt{n} c_{01} , \end{aligned} \tag{28}$$

where $t'_{n-1}(\delta_{1i})$ is the non-central t distribution with $n-1$ degrees of freedom and non-central parameter δ_{1i}.

Analogously, the p-value of C_{pui} is,

$$\begin{aligned} p\text{-value} &= P\left(\hat{C}_{pui} > w_{pui} | C_{pui} = c_{01}\right) \\ &= P\left(t'_{n-1}(\delta_{2i}) > 3\sqrt{n} w_{pui}\right) , \quad \delta_{2i} = 3\sqrt{n} c_{01} . \end{aligned} \tag{29}$$

If the p-value $\leq \alpha/k$, then H_0 can be rejected and this process is capable.

Similarly, the following statistical hypotheses concerning C_{pp} are considered. The process is of the required quality if $C_{pp} < c_{02}$, and is not if $C_{pp} > c_{02}$. The typically used benchmark values, c_{02}, such as the preset value of C_{pp} in Tables 4 or 5, are chosen for testing.

$$\begin{aligned} &H_0 : C_{ppi} \geq c_{02} \\ &H_1 : C_{ppi} < c_{02} . \end{aligned} \tag{30}$$

The original definition of C_{ppi} is,

$$C_{ppi} = \frac{(\mu_{Zi} - Z_i)^2}{D_i^2} + \frac{\sigma_{Zi}^2}{D_i^2} . \tag{31}$$

Thereby, from Eqs. 31 and 18,

$$\frac{(n-1)(n+\lambda_i)}{n} \times \frac{\hat{C}_{ppi}}{C_{ppi}} \sim \chi_n'^2(\lambda_i) , \quad \lambda_i = n(\mu_{Zi} - T_i)^2/\sigma_{Zi}^2 . \tag{32}$$

Boyles [4] suggests approximating the distribution, $\chi_n'^2(\lambda_i)$, introduced by Patnaik [16]. That is,

$$\chi_n'^2(\lambda_i) \approx e_i \chi_{\nu_i}^2 , \tag{33}$$

where $\chi_n'^2(\lambda_i)$ is the non-central χ^2 distribution with n degrees of freedom and non-central parameter $\lambda_i = n(\mu_{Zi} - T_i)^2/\sigma_{Zi}^2$, $e_i = (n+2\lambda_i)/(n+\lambda_i)$ and $\nu_i = (n+\lambda_i)^2/(n+2\lambda_i)$.

Then,

$$\frac{(n-1)\nu_i}{n} \times \frac{\hat{C}_{ppi}}{C_{ppi}} \sim \chi^2(\nu_i) , \tag{34}$$

where $\chi^2(\nu_i)$ is the χ^2 distribution with ν_i degrees of freedom. Suppose the observed value of the test statistic $\hat{C}_{ppi} = w_{ppi}$. Then,

$$\begin{aligned} p\text{-value} &= P\left(\hat{C}_{ppi} < w_{ppi} | C_{ppi} = c_{02}\right) \\ &= P\left(\chi_{\nu_i}^2 < \frac{(n-1)\nu_i}{n} \times \frac{w_{ppi}}{c_{02}} \middle| C_{ppi} = c_{02}\right) . \end{aligned} \tag{35}$$

If the p-value $\leq \alpha/k$, then H_0 can be rejected and this process is capable. At this juncture, the degrees of freedom ν_i of the

Table 6. An evaluating checklist of process performance with k quality characteristics

Index	Specification *LSL*	Specification *USL*	Sample mean	Sample *STD*	Preset value	Estimate value	p-value	Comment
C_{pl1}	L_{X1}		$\bar{X}_1$	S_{X1}	c_{01}	w_{pl1}	p_{pl1}	
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
C_{pli}	L_{Xi}		$\bar{X}_i$	S_{Xi}	c_{01}	w_{pli}	p_{pli}	
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
C_{pla}	L_{Xa}		$\bar{X}_a$	S_{Xa}	c_{01}	w_{pla}	p_{pla}	
C_{pu1}		U_{Y1}	$\bar{Y}_1$	S_{Y1}	c_{01}	w_{pu1}	p_{pu1}	
$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
C_{pui}		U_{Yi}	$\bar{Y}_i$	S_{Yi}	c_{01}	w_{pui}	p_{pui}	
$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
C_{pub}		U_{Yb}	$\bar{Y}_b$	S_{Yb}	c_{01}	w_{pub}	p_{pub}	
C_{pp1}	L_{Z1}	U_{Z1}	$\bar{Z}_1$	S_{Z1}	c_{02}	w_{pp1}	p_{pp1}	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
C_{ppi}	L_{Zi}	U_{Zi}	$\bar{Z}_i$	S_{Zi}	c_{02}	w_{ppi}	p_{ppi}	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
C_{ppc}	L_{Zc}	U_{Zc}	$\bar{Z}_c$	S_{Zc}	c_{02}	w_{ppc}	p_{ppc}	

chi-square distribution may or may not be an integer. Here, an approximate chi-square value can be obtained by interpolating values according to the chi-square distribution.

Finally, the p-value of the estimators of C_{pl}, C_{pu} and C_{pp} are used to assess the process potential and performance for complete products, and are summarized in Table 6.

5 Test procedure and example

This section describes the use of the table presented in the preceding section. The application of the presented procedure to assess process performance and judge whether a process is capable given k quality characteristics, is demonstrated.

Step 1: For given k and p, determine the preset value of indices c_0 and c_{01} for k quality characteristics, using Tables 4 and 5. Determine also the α-risk (normally set to 0.01, 0.025 or 0.05), which is the probability of incorrectly concluding that a process is incapable.

Step 2: Determine the estimated values of $\hat{C}_{pli}$, $\hat{C}_{pui}$, and $\hat{C}_{ppi}$ from the sample and the corresponding values w_{pli}, w_{pui}, and w_{ppi}.

Step 3: Find the p-values that correspond to w_{pli}, w_{pui} or w_{ppi} for sample size n.

Step 4: (a) If p-value $> \alpha/k$, for some $\hat{C}_{pli} \geq w_{pli}$, $\hat{C}_{pui} \geq w_{pui}$ or $\hat{C}_{ppi} \leq w_{ppi}$, then perform some quality-improving tasks. Insert "***" in the ith cell of the comment column in Table 6.

(b) If p-value $\leq \alpha/k$, for some $\hat{C}_{pli} \geq w_{pli}$, $\hat{C}_{pui} \geq w_{pui}$ or $\hat{C}_{ppi} \leq w_{ppi}$, then the ith characteristic is of the required quality, and the ith cell in the comment column is left blank in Table 6.

Step 5: The process is capable if p-value $\leq \alpha/k$, for all $\hat{C}_{pli} \geq w_{pli}$, $\hat{C}_{pui} \geq w_{pui}$ or $\hat{C}_{ppi} \leq w_{ppi}$, that is, all cells in the comment column are blank in Table 6.

An example is used to explicate the procedure presented here. This example concerns the process of producing a complete product. These processes are mutually independent and have three quality characteristics A ("smaller-the-better"), B and C ("nominal-the-best"), for which the data is shown in Table 7.

The process data yield,

$$\hat{C}_{pu} = 2.392157\ , \quad \hat{C}_{pp1} = 0.009586\ , \quad \text{and} \quad \hat{C}_{pp2} = 1.020027\ .$$

1. Assume that p_i represents the ith quality characteristic of the process yield, $i = 1, 2, 3$, and a complete product must be of a quality of at least $1-\alpha = p = 99.73\%$. If $p_1 = p_2 = p_3$, then the %Yield of each characteristic of a process for producing entire product equals or exceeds $p_i = 0.9973^{1/3} = 0.9990992$, $i = 1, 2, 3$, and $n = 30$, $k = 3$.
   Finally, the preset values of indices $c_{01} = 1.040365 = (1/3)\Phi^{-1}((0.9973)^{1/3})$ by Eq. 23, for C_{pu}, and $c_{02} = (3/\Phi^{-1}(((0.9973)^{1/3}+1)/2))^2 = 0.8165811$ by Eq. 25, for C_{pp} are set.
2. $\delta = 3 \times \sqrt{30} \times 1.040365 = 17.094940$, $\lambda_1 = 30$, $\lambda_2 = 0.10625457$, and $\nu_1 = 40$, $\nu_2 = 30.0003737$, yielding $3\sqrt{n}w_{pu} = 3 \times \sqrt{30} \times 2.392157 = 39.3071834$.
   The following p-values are obtained.

$$\begin{aligned}
p\text{-value of } C_{pu} &= P(t'_{29}(17.094940) > 39.3071834) \\
&= 0.0000\ , \\
p\text{-value of } C_{pp1} &= P\left(\chi^2_{40} < \frac{29 \times 40}{30} \times \frac{0.009586}{0.8165811}\right) \\
&= P\left(\chi^2_{40} < 0.453915\right) = 0.0000\ , \\
p\text{-value of } C_{pp2} & \\
= P&\left(\chi^2_{30.0003737} < \frac{29 \times 30.0003737}{30} \times \frac{1.020027}{0.8165811}\right) \\
&= P\left(\chi^2_{30.0003737} < 36.2256137\right) \\
&= 0.2008\ .
\end{aligned}$$

3. Table 8 summarizes these results.

Table 7. Data for process of producing a complete product, and process benchmarks, c' $(= c_{01}, c_{02})$

Quality characteristic	LSL	USL	$\bar{x}_i$	S_i	n_i	$T_i = m_i$	c'
$A - C_{pu}$	–	24	17.9	0.85	30	12	1.040365
$B - C_{pp1}$	8.24	8.76	8.494	0.006	30	8.5	0.8165811
$C - C_{pp2}$	−5	5	0.1	1.6803	30	0	0.8165811

Table 8. Evaluation checklist of process performance for three quality characteristics

Index	LSL	USL	$\bar{x}_i$	S_i	c'	Estimate value	p-value	Comment
$A - C_{pu}$	–	24	17.9	0.85	1.040365	$\hat{C}_{pu} = 2.392157$	0.0000	
$B - C_{pp1}$	8.24	8.76	8.494	0.006	0.8165811	$\hat{C}_{pp1} = 0.009586$	0.0000	
$C - C_{pp2}$	−5	5	0.1	1.6803	0.8165811	$\hat{C}_{pp2} = 1.020027$	0.2008	***

Table 8 shows that the processes quality characteristics A and B are of the required quality, and meet the consumers' expectations. However, the p-value of $C_{pp2} = 0.2008 > \alpha/k = 0.0027/3 = 0.0009$, shows that the quality characteristic C is "inadequate", and requires improvement, until the p-value of $C_{pp2} < \alpha/k$. Furthermore, the values of (C_{ia}, C_{ip}) are (0.004793, 0.004793) for B and (0.0036, 1.016427) for C. The separated information clearly distinguishes these processes.

Additionally, according to Greenwich and Jahr-Schaffrath [11], the conforming output proportions (COP) of quality characteristics B and C can be determined by computing $C_{cop} = 0.070866$ and $C_{cop} = 1.028755$, respectively. $C_{cop} = 0.070866 < 1$ for B, indicating a conforming output proportion far greater than 0.9973 and meeting the specifications, assuming normality. $C_{cop} = 1.028755 > 1$ for C, because its conforming output proportion is much less than 0.9973 and the process variation is greater. The specification is not met and the process variation must be reduced stepwise until the consumers' required quality is achieved.

6 Conclusion

Manufacturing industries to measure quantitatively process potential and performance have used capability indices Cp, Cpl, Cpu, Cpk and Cpp. Process capability analysis is a powerful tool for elucidating the ability of a process to manufacture products that meet specifications.

Bothe [15] stated that a customer purchases a complete product that consists of many characteristics. The customer's main concern is that all features are within specifications, so that the product meets his or her expectations. Currently, several methods exist for measuring the process capability of a single specification of a product. Such measures apply on to individual characteristics of a product.

This study developed a novel procedure of integrated analysis of the process capability for a complete product. The proposed p-value of the estimated index yields a user checklist that can be used not only to check whether the process is controlled but also to detect whether the product is defective in any quality characteristic.

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